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Brief paper

On the V-stability of complex dynamical networks[☆]

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Abstract

This paper introduces the concept of Lyapunov V-stability for complex dynamical networks. Under the new framework, each dynamical node is associated with a passivity degree, which indicates to what extent an effort is required for stabilizing the node. From this approach, the network stability problem is converted to measuring the negative definiteness of one simple matrix that characterizes the topology of the network. Pinning control is then suggested and investigated based on the new V-stability formulation. As an illustrative example, a network with different node dynamics and non-uniform coupling strengths is simulated to verify the analytic results. Moreover, a comparison study for three different kinds of networks is provided to further illustrate the novelty and efficacy of the proposed V-stability criterion and stabilization scheme.

Keywords: V-stability; Passivity; Complex network; Pinning control

1. Introduction

Various large-scale and complicated systems can be modeled by complex networks, including the Internet, WWW, genetic networks, social networks, and many others. The most remarkable recent advances are the developments of the small-world network model (Watts & Strogatz, 1998) and scale-free network model (Barabási & Albert, 1999), which have been shown to be closer to most real-world networks as compared with the classical random-graph model (Erdös & Rényi, 1959). Significant progress has been made in uncovering static structural properties of these two types of networks as summarized in the recent reviews (Albert & Barabási, 2002; Dorogovtsev & Mendes, 2002; Newman, 2003).

The dynamical behaviors of complex networks have become a focal topic of great interest in recent years, particularly the synchronization phenomena in various networks, from regular networks such as chains, grids and lattices (Parekh, Parthasarathy, & Sinha, 1998; Wu, 2002; Wu & Chua, 1995), to small-world and scale-free networks (Barahona & Pecora, 2002; Kocarev & Amato, 2005; Wang & Chen, 2002b, c). On the other hand, control of complex networks has also been studied, e.g., pinning control (Li, Wang, & Chen, 2004; Wang & Chen, 2002a). Differing from general dynamical systems, the behavior of a complex dynamical network is determined not only by the dynamical rules governing the isolated nodes, referred to as self-dynamics hereafter, but also by the information flows traveling along the links, which depend on the topology of the network. It turns out to be non-trivial to analyze the stability of a complex dynamical network due to the combining effects of the self-dynamics of the individual nodes and the interconnected network topology.

Starting from the familiar stability analysis of node self-dynamics and then taking into account the network structural effects, this study presents a new stability analysis scheme, named Lyapunov V-stability, or simply V-stability, for complex dynamical networks. Briefly, a common Lyapunov function $V(x)$ for all nodes in the network is constructed such that the self-dynamics of each node can be described by a passivity degree, a scalar parameter indicating the extent of the

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effort needed to stabilize the node in the direction that makes the derivation of $V(x)$ negative. Under the proposed *V*-stability scheme, a characteristic matrix for the stability of the network is derived in terms of the sum of two matrices: one denotes the passivity degree and the other, the network topology. Despite the conservativeness of the *V*-stability criterion, similar to general Lyapunov stability theories, the derived stability condition and the associate region of attraction are quite useful for network stabilization, such as the pinning control problem to be further studied in the sequel.

The rest of the paper is organized as follows. The next section describes the stability problem of complex dynamical networks, following which Section 3 further elaborates the main results of the paper, consisting of the *V*-stability scheme and the construction of a quadratic $V(x)$ and a general $V(x)$. Pinning control problem under the *V*-stability scheme is then investigated in Section 4, where a necessary condition for network stabilization in terms of the *V*-stability is derived. Some simulation results are reported in Section 5, on a simple network model of four different kinds of node self-dynamics in a non-uniform coupling configuration, and on a comparison of the stabilizability problem for three different kinds of networks, i.e., regular lattices, small-world networks and random networks. Finally, Section 6 concludes the paper.

2. Problem description

Consider a network consisting of N linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f_i(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij}a_{ij}\Gamma(x_j - x_i), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ are the state vectors of node i, $f_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ represents the self-dynamics of node i, constants c_{ij} are the coupling strengths between node i and node j, $i, j = 1, 2, \ldots, N$. In this model, the constant matrix $\Gamma \in \mathbb{R}^{n\times n}$ describes the way of linking the components in each pair of connected node vectors $(x_j - x_i)$, while the coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ denotes the coupling configuration of the entire network: if there is a connection between node i and node j, then $a_{ij} = a_{ji} = 1$; else $a_{ij} = a_{ji} = 0$. Define the degree k_i of node i as the number of connections at node i, satisfying

$$k_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ji}, \quad i = 1, 2, \ldots, N.$$

Furthermore, let the diagonal elements be $a_{ii} = -k_i$, $i = 1, 2, \ldots, N$, which means diffusive coupling, and define $c_{ii} = -(1/k_i)\sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij}a_{ij}$ for normalization. Then, network (1) can be rewritten in a compact form as

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j, \quad i = 1, 2, \ldots, N. \tag{2}$$

Assume that the network nodes have a common equilibrium state $\bar{x} \in \mathbb{R}^n$, satisfying

$$f_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, N. \tag{3}$$

Then, the homogeneous stationary state

$$x_1 = x_2 = \cdots = x_N = \bar{x} \tag{4}$$

is a trivial equilibrium point of the entire network (2), denoted by $\overline{X} = (\bar{x}^{\mathrm{T}}, \bar{x}^{\mathrm{T}}, \ldots, \bar{x}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}$. The objective is then to investigate the stability problem of the network about $\overline{X}$.

Assume also that the functions f_i are differentiable at $\bar{x}$. Then, linearizing (2) yields

$$\dot{Y} = (\mathscr{J} + G \otimes \Gamma)Y, \tag{5}$$

where $Y = (x_1^{\mathrm{T}} - \bar{x}_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}} - \bar{x}_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}$, $\mathscr{J} = \mathrm{diag}(J_1, \ldots, J_n) \in \mathbb{R}^{nN\times nN}$ with Jacobian matrices $J_i = \partial f_i(\bar{x})/\partial x$, and $G = (g_{ij}) \in \mathbb{R}^{N\times N}$ with $g_{ij} = c_{ij}a_{ij}$, $i, j = 1, 2, \ldots, N$, and $\otimes$ is the Kronecker product notation. System (5) is a linear autonomous system, so according to the Lyapunov indirect method, $\overline{X}$ is asymptotically stable if all the eigenvalues of the matrix $(\mathscr{J} + G \otimes \Gamma)$ are located on the left-half plane; otherwise, it is unstable or undetermined (all eigenvalues are located on the imaginary axis).

It is well known that if the linearized system (5) is used for analysis, only local stability results can be obtained in general. Moreover, the following problems are observed:

- The dimension of $(\mathscr{J} + G \otimes \Gamma)$ is the product of the order of nodes and the number of nodes in the network, which will become very large as the size of the network or the order of nodes increases.
- Except for the case of linear node self-dynamics, the linearization method only works in a neighborhood of $\overline{X}$, which is an existing but unknown local domain, usually small, therefore difficult to use.
- The influences of both node self-dynamics and network topology on the network stability cannot be estimated from the linearized system (5).

Therefore, this paper will bypass the commonly used linearized system (5) but turns to seek for global stability results based on the original network (1) or (2).

3. Main results

First, the concept of *V*-stability is introduced.

3.1. V-stability

The following assumption is needed throughout the paper, where

$$D_i = \{x_i : \|x_i - \bar{x}_i\| < \alpha\}, \quad \alpha > 0, \quad D = \bigcup_{i=1}^{N} D_i.$$

Assumption 1. There is a continuously differentiable Lyapunov function $V(x) : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}_+$ satisfying $V(\bar{x}) = 0$

with $\bar{x} \in D$, such that for each node function $f_i(x_i)$, there is a scalar θ_i guaranteeing

$$\frac{\partial V(x_i)}{\partial x_i}(f_i(x_i) - \theta_i \Gamma(\bar{x} - x_i)) < 0$$
$$\forall x_i \in D_i, \quad x_i \neq \bar{x}, \; i = 1, 2, \ldots, N. \tag{6}$$

Here θ_i will be called *passivity degree* in the following. It is important to note that, since θ_i are not unique in general, the theoretical passivity degree is defined to be the largest one; however, since this quantity is only used to derive sufficient conditions for the stability in association with the common Lyapunov function V, any of such θ_i values can be taken as a passivity degree in practice.

Many complex dynamical networks satisfy this assumption including, for example, a linearly coupled array of Chua's oscillators (Wu & Chua, 1995), a lattice of chaotic lasers (DeShazer, Breban, Ott, & Roy, 2001), a small-world network of x-coupled Rössler chaotic oscillators (Barahona & Pecora, 2002), and the examples to be further discussed in Section 5 below. The physical meaning of this assumption is that there exists a common energy-like function V for all nodes in the network, such that the energy of V is decreasing along the evolution of node dynamics, with the passivity degree θ_i being positive when its connected nodes have been forced evolving to the equilibrium point. Intuitively, $\theta_i < 0$ means that the ith node needs energy from outside to become stable, while $\theta_i > 0$ means that the ith node itself is already stable; therefore, can provide some extra energy to stabilize other nodes in the networks.

Example 1. Consider a network consisting of two kinds of node self-dynamics, the Lorenz system $L(x)$ (Lorenz, 1963) and the Chen system $C(x)$ (Chen & Ueta, 1999), with matrix $\Gamma = \mathrm{diag}(0, 1, 1)$, where

$$L(x): \begin{cases} \dot{x}_1 = a_L(x_2 - x_1), \\ \dot{x}_2 = c_L x_1 - x_2 + x_1 x_3, \\ \dot{x}_3 = x_1 x_2 + b_L x_3, \end{cases} \tag{7}$$

$$C(x): \begin{cases} \dot{x}_1 = a_C(x_2 - x_1), \\ \dot{x}_2 = (c_C - a_C)x_1 + c_C x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - b_C x_3. \end{cases} \tag{8}$$

When $a_L = 10$, $b_L = \frac{8}{3}$ and $c_L = 28$, the Lorenz system is in a chaotic state, so is the Chen system with $a_C = 35$, $b_C = 3$ and $c_C = 28$. It can be easily verified that there is a common Lyapunov function, $V = x^{\mathrm{T}} P x$ with $P = \mathrm{diag}(0.1, 1, 1)$, such that the passivity degrees of $L(x)$ and $C(x)$ are $\theta_L = -29$ and $\theta_C = -2$, respectively, with respect to the trivial equilibrium $\bar{x} = 0$.

Now, without loss of generality, assume $\overline{X} = 0$ and consider the following Lyapunov function for the whole network (2):

$$V_N(X) = \sum_{i=1}^{N} V(x_i), \quad X = (x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}})^{\mathrm{T}}. \tag{9}$$

Its time derivative along trajectory X is given by

$$\dot{V}_N(X) = \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} f_i(x_i) + \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i). \tag{10}$$

It is easy to see that $V_N(\overline{X}) = 0$ and $\dot{V}_N(\overline{X}) = 0$. Also, Assumption 1 implies that, for $X \neq \overline{X} = 0$, one has

$$\dot{V}_N(X) < -\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \theta_i \Gamma x_i + \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i). \tag{11}$$

Rewrite inequality (11) as

$$\dot{V}_N(X) < M(X),$$

where

$$M(X) = \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \left(-\theta_i \Gamma x_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i) \right). \tag{12}$$

The following result can be easily verified, where

$$\mathscr{D} = D_1 \times D_2 \times \cdots \times D_N \subseteq \mathbb{R}^{nN}.$$

Lemma 1. (i) *Network* (2) *is locally asymptotically stable about its equilibrium point if* $M(X) \leqslant 0$ *for all* $X \in \mathscr{D} \backslash \{0\}$.

(ii) *Network* (2) *is locally exponentially stable about its equilibrium point if* $M(X) \leqslant -\mu_1 \|X\|^2$, $\mu_2 \|x\|^2 \leqslant V(x) \leqslant \mu_3 \|x\|^2$ *for some constants* $\mu_1, \mu_2, \mu_3 > 0$ *for all* $X \in \mathscr{D}$.

Moreover, the region of attraction is given by

$$\Omega = \{X : V_N(X) < r\} \tag{13}$$

with $r = \inf_{X \in \partial \mathscr{D}} V_N(X)$. *In the case of* $\mathscr{D} = \mathbb{R}^{nN}$, *the above stability becomes global.*

Three remarks are in order.

First, the node functions f_i $(i = 1, 2, \ldots, N)$ do not appear in formula (12) in which they are replaced by the corresponding passivity degree θ_i, respectively. Thus, the impact of the node self-dynamics on the network stability is shifted to the passivity degree θ_i.

Second, if $(\partial V(x_i)/\partial x_i)\Gamma x_i > 0$, then it can be concluded that the ith node having a positive passivity degree will be dissipative when it is being isolated.

Third, all the above derivations are based on Assumption 1, i.e., there exists a common Lyapunov function $V(x)$, which usually is undetermined and non-unique. Hence, the above proposed method of converting the original stability problem to the study of the negativity property of the function $M(X)$ strongly depends on the selection of $V(x)$ (and of θ_i). For this reason, the stability so derived is called the *V-stability*.

3.2. Quadratic function for $V(x)$

First, consider the case where the common Lyapunov function $V(x)$ is selected to be a quadratic monomial; namely, there is a symmetric and positive definite matrix Q such that $V(x) = x^{\mathrm{T}}Qx$.

Theorem 2. *Suppose that there exists a function* $V(x)=x^{\mathrm{T}}Qx$, *with* Q *being a symmetric and positive definite matrix, satisfying Assumption* 1 *with passivity degree values* θ_i, *such that the following inequality holds*:

$$Q\Gamma + \Gamma^{\mathrm{T}}Q \geqslant 0. \tag{14}$$

Then, network (2) *is V-stable if the following inequality is satisfied*:

$$-\Theta + G \leqslant 0, \tag{15}$$

where $\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^{N\times N}$ *and* $G = (g_{ij}) = (c_{ij}a_{ij}) \in \mathbb{R}^{N\times N}$. *Moreover, if* $\mathscr{D} = \mathbb{R}^{nN}$, *then the above stability is global.*

Proof. Since $\partial V(x)/\partial x = 2x^{\mathrm{T}}Q$, the function $M(X)$ defined in (12) can be written as

$$M(X) = \sum_{i=1}^{N} x_i^{\mathrm{T}}\left(-\theta_i Q\Gamma x_i + \sum_{j=1}^{N} c_{ij}a_{ij}Q\Gamma x_j\right), \tag{16}$$

or, in the Kronecker product form,

$$M(X) = X^{\mathrm{T}}(-\Theta + G) \otimes Q\Gamma X. \tag{17}$$

By (14), one can easily show that the matrix $(-\Theta + G) \otimes Q\Gamma$ is negative definite (Lütkepohl, 1996). The proof is thus completed. □

Theorem 2 gives a sufficient condition (15) for the V-stability of network (2) about its equilibrium point. This condition is simple, which only consists of the information about the node self-dynamics described by parameters Θ and the information about the network topology G. Here, G is negative semi-definite, since it is a symmetric matrix with zero row-sums and non-negative off-diagonal elements.

Corollary 3. *Suppose that there is a function* $V(x) = x^{\mathrm{T}}Qx$, *with* Q *being a symmetric and positive definite matrix, satisfying inequality* (14) *and Assumption* 1 *with positive passivity degree values* $\theta_i > 0$. *Then, network* (2) *is V-stable.*

It can be seen that the smaller the passivity degree values θ_i, the more likely that condition (15) is violated.

It can also be seen that if all the self-dynamics are the same and the function is continuously differentiable at the equilibrium point (here, the origin), then the best way to select Q is to solve the following optimization problem:

$$\max_{Q} \quad \alpha$$
$$\text{subject to:} \quad Q(J+\alpha\Gamma) + (J+\alpha\Gamma)^{\mathrm{T}}Q \leqslant 0,$$
$$Q\Gamma + \Gamma^{\mathrm{T}}Q \geqslant 0, \tag{18}$$

where J is the corresponding Jacobian matrix of the nodes.

This optimization problem with strict inequalities can be easily solved by the well-known LMI toolbox (Gahinet, Nemirovski, Laub, & Chilal, 1995).

3.3. General functions for $V(x)$

In a more general case, a Lyapunov function may not be a quadratic monomial. This case is now discussed.

Theorem 4. *Suppose that there is a nonlinear function* $V(x)$, *satisfying Assumption* 1 *with passivity degree values* θ_i, *and that there are two positive constants* d_1 *and* d_2 *such that the following inequalities are satisfied*:

$$\left(\frac{\partial V(x_i)}{\partial x_i}\right)\Gamma x_i \leqslant d_1\|x_i\|^2, \tag{19}$$

$$\left(\frac{\partial V(x_i)}{\partial x_i} - \frac{\partial V(x_j)}{\partial x_j}\right)\Gamma(x_i - x_j) \geqslant d_2\|x_i - x_j\|^2. \tag{20}$$

Furthermore, define a diagonal matrix $\Phi = \mathrm{diag}\{\phi_i\} \in \mathbb{R}^{N\times N}$ *with* $\phi_i = -\theta_i d_1$ *if* $\theta_i \leqslant 0$ *or else* $\phi_i = -\theta_i d_2$. *If the following inequality is satisfied*:

$$\frac{1}{d_2}\Phi + G \leqslant 0, \tag{21}$$

then network (2) *is V-stable. Moreover, if* $\mathscr{D} = \mathbb{R}^{nN}$, *then the above stability is global.*

Proof. Note that $c_{ij}a_{ij} = c_{ji}a_{ji}$ and $c_{ii}a_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}a_{ij}$, $i, j = 1, 2, \ldots, N$. It follows from (20) that formula (12) can be rewritten as

$$\begin{aligned} M(X) = {} & \sum_{i=1}^{N} -\left(\frac{\partial V(x_i)}{\partial x_i}\right)\theta_i\Gamma x_i \\ & - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_{ij}a_{ij}\left(\frac{\partial V(x_i)}{\partial x_i} - \frac{\partial V(x_j)}{\partial x_j}\right) \\ & \times \Gamma(x_i - x_j). \end{aligned} \tag{22}$$

Note also that inequality (20) means that $(\partial V(x_i)/\partial x_i)\Gamma x_i \geqslant d_2\|x_i\|^2$, so that

$$-\left(\frac{\partial V(x_i)}{\partial x_i}\right)\theta_i\Gamma x_i \leqslant \phi_i\|x_i\|^2, \quad i = 1, 2, \ldots, N. \tag{23}$$

Applying (20) and (23) yields

$$M(X) \leqslant Z^{\mathrm{T}}(\Phi + d_2 G)Z, \tag{24}$$

where $Z = (\|x_1\|, \|x_2\|, \ldots, \|x_N\|)^{\mathrm{T}} \in \mathbb{R}^N$.

Now, by inequality (21), the right-side of (24) is non-positive for all Z. This means that network (2) is V-stable. The proof is thus completed. □

Corollary 5. *If that there is a nonlinear function $V(x)$ satisfying the two inequalities* (19)–(20) *and Assumption* 1 *with positive passivity degree values $\theta_i > 0$, then network* (2) *is V-stable.*

In view of the main results derived in this section, the stability problem of a complex dynamical network can be converted, in a sense simplified, to determining whether the sum of two matrices is negative definite. This is the V-stability scheme proposed, in which inequalities (15) and (21) are convenient to use, which; therefore, are called the *characteristic matrices* of the network. The physical meaning of the derived condition is clear: the first matrix shows the impact of the node self-dynamics and the second matrix indicates the influence of the network topology on the network stability.

Note that the Lyapunov function (9) is a special one in the class of Lyapunov functions used for interconnected systems in Araki (1978) and Ikeda and Šiljak (1981), where, however, the Lyapunov function $V(x_i)$ for the subsystem is restricted to quadratic functions. Along the same line, we can easily get the stability results with more design freedom by using the Lyapunov function suggested in Araki (1978) and Ikeda and Šiljak (1981) for a quadratic $V(x_i)$, but it can be found that with more design freedom results it is also more complex than (15) herein and therefore more difficult to apply.

4. Pinning control

In general, if a given network is unstable, then stabilizing it by means of control is an important problem for study. It follows from Corollaries 3 and 5 that a simple way to stabilize a network is to place a suitable controller at every node such that each has a positive passivity degree value. This, however, is clearly impractical. In Wang and Chen (2002a) and Li et al. (2004), the pinning control strategy was applied to scale-free networks, where only a small fraction of nodes are applied with pinning control. Under the V-stability scheme, developed above, pinning control can be thought of as a strategy that can make the passivity degree values of the controlled nodes become positive and as large as possible.

Here, without loss of generality, assume that the first m nodes are controlled with pinning controllers of the form

$$u_i = -K_i x_i, \quad i = 1, 2, \ldots, m, \tag{25}$$

where the gain matrices $K_i \in \mathbb{R}^{l\times n}$ are to be designed. Thus, the self-dynamics of the controlled nodes become

$$\dot{x}_i = f_i(x_i) - B_i K_i x_i, \quad i = 1, 2, \ldots, m, \tag{26}$$

where $B_i \in \mathbb{R}^{n\times l}$ is the input matrix of each control node. Consequently, for the controlled network, one has

$$\frac{\partial V(x_i)}{\partial x_i}(f_i(x_i) + \theta_i \Gamma x_i - B_i K_i x_i + \kappa_i \Gamma x_i) < 0 \quad \forall x_i \in D_i \subseteq D, \quad x_i \neq 0 \tag{27}$$

with constants $\kappa_i \geqslant 0$. The passivity degree values of the controlled nodes are increased by a factor of κ_i.

For simplicity, only the case with a quadratic function $V(x)$ is discussed here. In this case, the characteristic matrix of the controlled network is given by

$$C = -\Theta + G - \mathscr{K}, \tag{28}$$

where $\mathscr{K} \in \mathbb{R}^{N\times N}$ is an diagonal matrix with m elements κ_i, $i = 1, 2, \ldots, m$, and its remaining $N - m$ elements are all zero.

Thus, under the V-stability scheme, the pinning control problem becomes determining how many and which nodes are chosen for pinning, subject to the condition that the characteristic matrix C is seminegative definite.

It is not yet easy to explicitly express the relation between the negative definiteness of C and the matrix $\mathscr{K}$. Nevertheless, from matrix theory, one can derive the following simple results.

Lemma 6. *If the coupling strength is sufficiently strong and κ_i, $i = 1, 2, \ldots, m$, is large enough, then C is negative definite with nonzero m.*

Proof. This is obvious, since $\mathscr{K}$ will be diagonally dominant, which completes the proof. □

Proposition 7. *If the characteristic matrix $-\Theta + G$ has m non-negative eigenvalues, and if the closed-loop characteristic matrix C is negative definite, then the number of nodes to be selected for control cannot be less than m.*

Proof. Suppose the opposite; namely, the number of controlled nodes is less than m. Then, by the construction of $\mathscr{K}$, one has

$$\operatorname{rank}(\mathscr{K}) < m. \tag{29}$$

Without loss of generality, the symmetric matrix $-\Theta + G$ can be written as

$$-\Theta + G = P \operatorname{diag}\{\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_N\} P^{\mathrm{T}} \tag{30}$$

with $P \in \mathbb{R}^{N\times N}$ being an orthogonal matrix, and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$. Note that $-\Theta + G \geqslant PWP^{\mathrm{T}}$, with $W = \operatorname{diag}\{0, \ldots, 0, \lambda_{m+1}, \ldots, \lambda_N\}$. Since $C < 0$, one has $PWP^{\mathrm{T}} + \mathscr{K} < 0$. However, by (29), one has $\operatorname{rank}(PWP^{\mathrm{T}})+\operatorname{rank}(\mathscr{K}) < n$, which is a contradiction to the fact that $PWP^{\mathrm{T}} + \mathscr{K}$ is negative definite. The proof is thus completed. □

Note that although the above two results were derived for the quadratic case, similar results can be easily obtained for the general case, where the signs of the passivity degree values of the controlled nodes may change.

5. Simulation study

5.1. A simple but non-trivial network example

To illustrate the proposed V-stability scheme, consider a simple network with four different nodes and five non-uniform

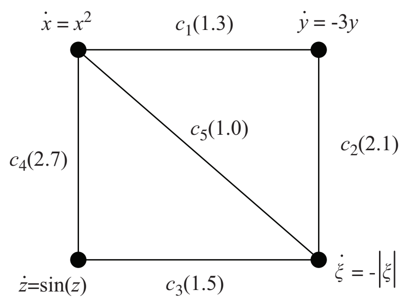

Fig. 1. The topological structure and self-dynamics of network (31).

links. The node self-dynamics are described by

$$\dot{x} = x^2, \quad \dot{y} = -3y, \quad \dot{z} = \sin z, \quad \dot{\xi} = -|\xi|, \tag{31}$$

and the coupling strengths are $c_1 = 1.3$, $c_2 = 2.1$, $c_3 = 1.5$, $c_4 = 2.7$ and $c_5 = 1.0$, respectively. Fig. 1 visualizes this network. Obviously, $\overline{X} = (\bar{x}, \bar{y}, \bar{z}, \bar{\xi})^{\mathrm{T}} = 0$ is the equilibrium point. It can be verified that Assumption 1 posed at the beginning of Section 3 is satisfied.

The objective is to study the V-stability of the network about $\overline{X}$, and to determine if the network is not stable then how to stabilize it by pinning control.

Note that since the node self-dynamics are different from each other, this network system is non-trivial one. The existing control methods such as those proposed in Wang and Chen (2002a) and Li et al. (2004) cannot be applied. Moreover, the function in the ξ node is not differentiable at the equilibrium $\xi = 0$, so any linearization method cannot be rigorously utilized here.

From the approach developed in this paper, take the common Lyapunov function $V(X)$ to be a very simple one:

$$V(X) = X^{\mathrm{T}} X, \tag{32}$$

where $X = [x, y, z, \xi]^{\mathrm{T}}$. Then, the passivity degree values of all nodes, except the node x, can be obtained directly, as $\theta_y = 2.999$, $\theta_z = -1.001$ and $\theta_\xi = -1.001$, over the entire domain of $\mathbb{R}$. For the node x, it can be easily seen that any scalar θ_x is feasible over the domain of $\{x : x < -\theta_x\}$. Since the region of attraction should contain the origin in this case, a qualified passivity degree value for the node x should satisfy $\theta_x < 0$. Therefore select $\theta_x = -\gamma$ with $\gamma > 0$. Thus, the corresponding feasible domains are $\mathscr{D}_x = \{x : |x| < \gamma\}$ and $\mathscr{D}_y = \mathscr{D}_z = \mathscr{D}_\xi = \mathbb{R}$. It then follows from definition (13) that the region of attraction is given by

$$\Omega = \{(x, y, z, \xi) : x^2 + y^2 + z^2 + \xi^2 < \gamma^2\}, \tag{33}$$

provided that the network is V-stable, which is verified below.

5.1.1. Stability analysis

To verify the V-stability, one needs to calculate the eigenvalues of the characteristic matrix. Here, assume that the coupling strengths are adjustable by a factor of d, in order to show the relation between the coupling strengths and the V-stability.

Thus, the characteristic matrix of the non-controlled network (31) can be written as

$$C_{\text{open}} = -\Theta + dG, \tag{34}$$

with $\Theta = \operatorname{diag}\{-\gamma, 2.999, -1.001, -1.001\}$. Arrange the eigenvalues of C_{open} by $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$. Since G is a symmetric and positive semi-definite matrix, it can be seen that the eigenvalues of C_{open} are non-decreasing as $-\gamma$ increases (Lütkepohl, 1996). Thus, in the worst case, one may take the limiting value $\gamma = 0$ and calculate the eigenvalues from $d = 0$ to 3, with step size 0.1, so as to obtain the numerical results shown in Fig. 2. When $d = 0$, there are two positive eigenvalues. This means that if all nodes are isolated, the entire network cannot achieve the V-stability. As d increases, all the eigenvalues decrease. Eventually, all the eigenvalues become negative when $d \geqslant 2.2$. With $d = 1$, giving the network shown in Fig. 1, it can be found that even in the case where the initial states are very close to the origin, the network is not stable. In simulations, at some points between $d = 2.1$ and 2.2, the network will evolve to be locally stable from being unstable. This is consistent with the analytic results obtained under the V-stability scheme.

Set $d = 3$. Then, it can be found by calculating the eigenvalues of C_{open} that network (31) can achieve V-stability if $\gamma \leqslant 0.2$. It then follows from (33) that network (32) is locally exponentially stable with the region of attraction being $\Omega = \{X : \|X\| \leqslant 0.2\}$. This result is quite conservative for network (31). However, as is well known, it is extremely difficult to exactly allocate the region of attraction for a complex network. From this point of view, the region of attraction found above is reasonable.

On the other hand, simulation shows that network (31) diverges from the initial state $X(0) = (0, 0, 10, 0)^{\mathrm{T}}$ but converges quickly to the origin from $X(0) = (10, -10, 10, -10)^{\mathrm{T}}$. To visualize the result, Fig. 3(a) shows the divergent phenomenon.

5.1.2. Pinning control

In the last subsection, one has seen that by using a three-time stronger coupling strength, the V-stability of network (31) can be ensured but only in a small region of attraction: $\Omega = \{X : \|X\| \leqslant 0.2\}$. The objective here is to enlarge this region of attraction by applying pinning control.

Consider again the divergent network (31) as shown in Fig. 3(a). In view of $X(0) = (0, 0, 10, 0)^{\mathrm{T}}$ and (33), it is very natural to enlarge θ_x by using $\gamma > 10$, say, $\gamma = 10.0001$, which can be reached by the following control law:

$$u(x) = -20.001x. \tag{35}$$

It can be easily verified that the closed-loop characteristic matrix is negative definite. Thus, all trajectories starting from the region of $\Omega = \{X : \|X\| \leqslant 10\}$ will exponentially converge to the origin. Fig. 3(b) shows the evolution of the network states.

If the coupling strength is weakened to $d = 0.1$, then two eigenvalues of C_{open} will become positive (see Fig. 2). By Proposition 7, two controllers are necessary to be used in order to make network (31) V-stable about the origin. Set the control

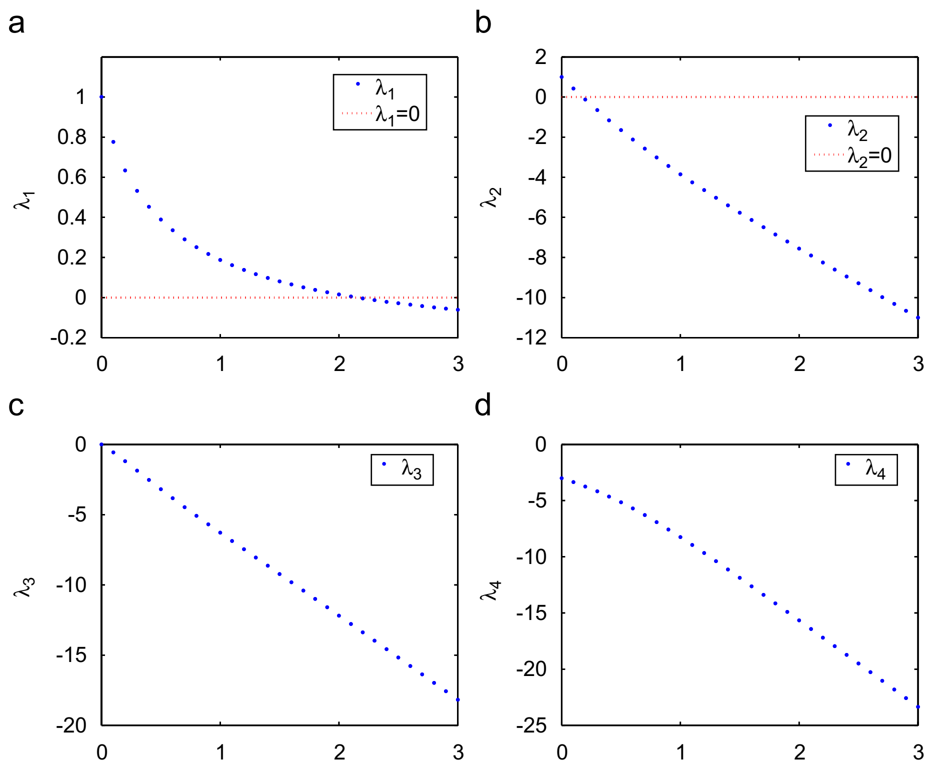

Fig. 2. The relation between the eigenvalues of the characteristic matrix and the coupling-strength factor d, varying from 0 to 3 in step size 0.1.

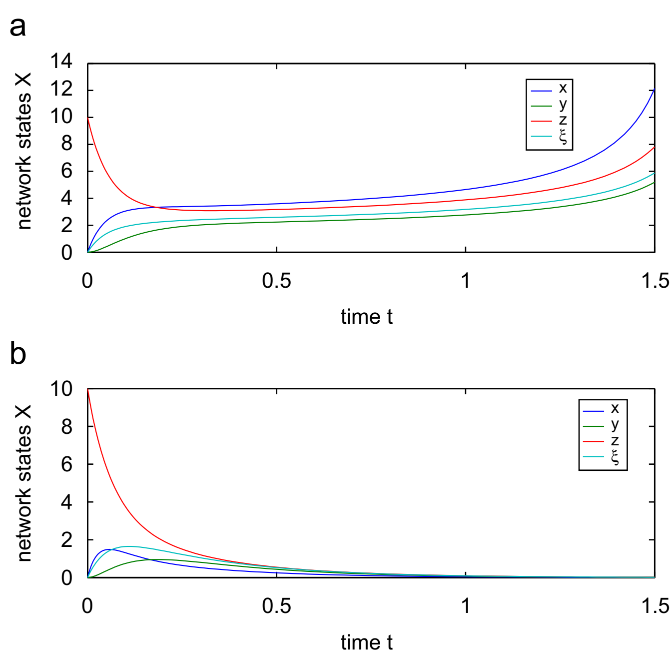

Fig. 3. The evolution of network states with initial state $X(0) = (0, 0, 10, 0)^{\mathrm{T}}$ and a three-time stronger coupling strength $3G$: (a) without control and (b) with pinning control (35).

law as

$$u_z = -z, \tag{36}$$

$$u_\xi = -\xi, \tag{37}$$

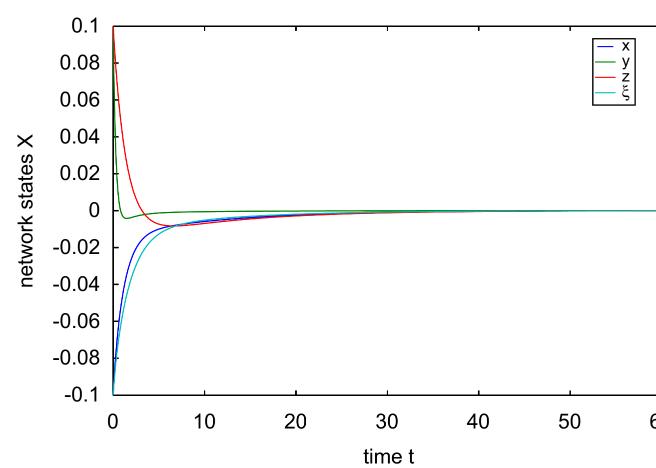

Fig. 4. The evolution of network states with initial state $X(0) = (-0.1, 0.1, 0.1, -0.1)^{\mathrm{T}}$ and coupling strength $0.1G$ under pinning controllers (36) and (37).

so as to obtain $\mathscr{K} = \mathrm{diag}(0, 0, -1, -1)$. Applying (28), one guarantees the closed-loop characteristic matrix C_{close} be negative definite for $\gamma \leqslant 0.2$. Thus, in the case of $d = 0.1$, any trajectory starting from the region $\{X : \|X\| \leqslant 0.2\}$ will exponentially converge to the origin under the pinning controllers (36) and (37). The simulation result is shown in Fig. 4.

Another simulation study (its results are not illustrated here due to space limitation) has also shown that the given network with coupling strength $0.1G$ cannot be stabilized by any pinning control if only one node is pinned, which verifies Proposition 7.

5.2. A comparison study on different networks

In order to further illustrate the proposed V-stability scheme, several common networks are simulated and compared here, regarding how many nodes are necessary to be controlled to achieve the V-stability of the networks.

Assume that all the nodes in each network are identical and there is a Lyapunov function $V(x)$ satisfying the inequalities (19) and (20) with positive scalars d_1 and d_2 such that Assumption 1 holds for the passivity degree $\theta < 0$. Denote the coupled matrix of a network by G. Thus, the characteristic matrix can be written as $C_{\mathrm{open}} = d_2 C$ with

$$C = -\frac{d_1}{d_2}\theta I_N + cG, \tag{38}$$

where N is the size of the network, c is the coupling strength and I_N is the identity matrix of dimension N. One can conveniently analyze the stability of the given network by examining the matrix (38) without any extra effort such as the consideration of node self-dynamics.

Now, the number of necessary controllers in different networks is discussed based on Proposition 7. Three kinds of common networks are considered, i.e., regular lattices, small-world networks and random networks, following the formulations presented in Watts and Strogatz (1998).

Start with a ring of N nodes, each connecting to its m nearest neighbors by undirected links. With probability p, rewire each link to a node chosen uniformly at random without duplicative links. When $p = 0$ the network is a regular lattice, $0 < p < 1$ a small-world network, and $p = 1$ a random network. Denote by n the number of necessary controllers used to stabilize a network. Define $\sigma = -\theta d_1/cd_2$ to be the weighted inverse of the coupled strength. The relation between n and σ for these three networks is studied, with the assumption that the coupling strength c varies in such a way that the change of σ is from 0.1 to 10, with step size 0.1.

The simulation results are shown in Fig. 5, where the small-world networks are produced with probability $p = 0.3$ and the trajectories are the average outputs of 40 random groups. It can be seen that the required number of controllers increases as the coupling strength decreases (as σ increases). When the coupling strength is strong, a random network is easiest to be stabilized and the regular lattice is most difficult. As the coupling strength decreases below the first phase transition point, the situation is reversed, namely, a regular lattice is easiest and a random network becomes most difficult. Further decreasing it to be below the second phase transition point, so that the coupling strength is very weak, the same situation as the strong coupling strength arises. This is an interesting observation slightly contradictive to one's intuition that the random networks and small-world networks are always much easier to be controlled due to the existence of some long-range links. This new observation may be qualitatively explained as follows.

The existence of some long-range links increases the influence of the controlled nodes on the other nodes such that the number of required controllers can be smaller for small-world networks and random networks than the regular ones. But this

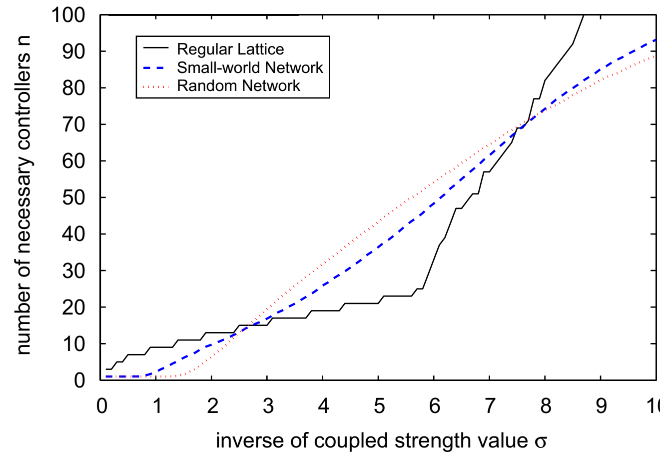

Fig. 5. The change of the number of necessary controllers as the coupling strength decreases. Three kinds of networks are compared (averaged outputs from 40 random groups).

happens under the condition that the coupling strength is large enough such that even the node with the smallest degree can become stable under the influence of the controlled nodes. As the coupling strength decreases, some nodes with small degrees cannot be stabilized indirectly by controlling their neighboring nodes. They require the controllers to stabilize them directly. Because the total number of links are constant, these kind of nodes exist mostly in random networks due to the averaged distribution of node degrees. The nodes in regular lattices are identical to each other, so that the number of required controllers is the smallest with the same coupling strength in comparison. This argument is also supported by the fact that the network trajectory varies only a little within an interval (of (3, 6) in the above example). As the coupling strength becomes weaker, all nodes cannot rely on the adjacent nodes to be stable. It can be seen from the simulations in Fig. 5 that a regular lattice quickly enters into the situation where all the nodes need to be controlled. However, the small-world networks and random networks allow some nodes with high degrees not be controlled. Therefore, after the second phase transition point, the regular lattices become again the most difficult ones to be stabilized.

6. Conclusions

In this paper, a new concept of Lyapunov V-stability has been introduced to the study of the asymptotic stability of complex dynamical networks. In this new setting, each dynamical node is associated with a passivity degree, which indicates to what extent an effort is needed to stabilize the node, and consequently to stabilize the entire network. From this new approach, the network stability problem is converted to determining the negative definiteness of one simple matrix that characterizes the topology of the network. To that end, pinning control has been suggested and investigated based on the new V-stability framework. The derived conditions are useful for determining how many nodes have to be pinned and how to enlarge the region of attraction by means of control. As an illustrative example, a network of four different nodes with five non-uniform coupling strengths has been simulated, which has verified the analytic results obtained in the paper. In addition, a comparison study for three different kinds of networks has been provided to further illustrate the novelty and efficacy of the proposed *V*-stability criterion and stabilization scheme.

It appears to be promising to further develop the notion of *V*-stability along with the concept of energy passivity in future studies of various complex dynamical networks, regarding in particular their stability, synchronizability, robustness, and optimal design. Towards this long-term goal, the present paper merely took up a first step, leaving many important tasks to be carried out in the near future.

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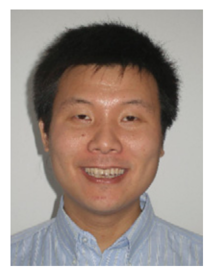

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